Inverse Reynolds-Dominance approach to transient fluid dynamics
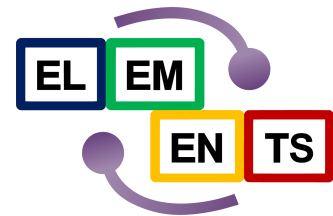
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Hydrodynamics: Conservation equations

$$\partial_\mu T^{\mu\nu} = 0 \,, \qquad \partial_\mu N^\mu = 0 \tag{1}$$

▶ Hydrodynamics: based on $(4 + 1 = 5)$ conservation equations
   - Ideal case: Sufficient (if equation of state is supplied)
     → Variables: ϵ, n, u^μ
   - Dissipative case: Underdetermined
     → Variables: ϵ, n, u^μ, Π, n^μ, $\pi^{\mu\nu}$

▶ **Fundamental question of dissipative hydrodynamics**: How to obtain information about the dissipative components of N^μ and $T^{\mu\nu}$?

Decomposition of conserved currents (Landau frame)

$$
\begin{aligned}
N^\mu &= n u^\mu + n^\mu \\
T^{\mu\nu} &= \epsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}
\end{aligned}
\tag{2}\tag{3}
$$

Projectors: $\Delta^{\mu\nu} := g^{\mu\nu} - u^\mu u^\nu$, $\Delta^{\mu\nu}_{\alpha\beta} := (\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha)/2 - \Delta^{\mu\nu}\Delta_{\alpha\beta}/3$

▶ First-order hydro: Relate dissipative quantities to fluid-dynamical gradients

$$\Pi = -\zeta\theta \ , \quad n^\mu = \kappa I^\mu \ , \quad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} \tag{4}$$

▶ (In standard hydrodynamic frame): **Acausal** and **unstable**!

▶ Second-order hydro: Treat dissipative quantites as dynamical, provide relaxation equations

Relaxation equations

$$
\begin{aligned}
\tau_\Pi\dot{\Pi} + \Pi &= -\zeta\theta + \text{h.o.t.} & \text{(5a)} \\
\tau_n\dot{n}^{\langle\mu\rangle} + n^\mu &= \kappa I^\mu + \text{h.o.t.} & \text{(5b)} \\
\tau_\pi\dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} &= 2\eta\sigma^{\mu\nu} + \text{h.o.t.} & \text{(5c)}
\end{aligned}
$$

▶ Needs input from **microscopic theory**

▶ This talk: Take **kinetic theory** as the foundation

■ Only consider the shear sector for simplicity

$$\theta := \partial^\mu u_\mu, \ \sigma^{\mu\nu} := \nabla^{\langle\mu}u^{\nu\rangle}, \ \nabla^\mu := \Delta^{\mu\nu}\partial_\nu, \ I^\mu := \nabla^\mu(\mu/T), \ A^{\langle\mu}B^{\nu\rangle} := \Delta^{\mu\nu}_{\alpha\beta}A^\alpha B^\beta$$

- ▶ Kinetic theory: describe evolution of one-particle distribution function $f(x, k)$ in phase space
  - ■ Split into local-equilibrium part $f_0(x, k)$ and deviation $\delta f(x, k)$
- ▶ Hydrodynamic quantities described through irreducible moments

$$\rho_r^{\mu_1 \cdots \mu_\ell}(x) := \int dK E_{\mathbf{k}}^r k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \delta f(x, k) \tag{6}$$

- ▶ Related through e.g. $\pi^{\mu\nu} \equiv \rho_0^{\mu\nu}$
- ▶ Dynamics described through Boltzmann equation $k^\mu \partial_\mu f(x, k) = C[f]$

Linearized moment equations

$$\sum_{n=0}^{N_2} \tau_{rn}^{(2)} \dot{\rho}_n^{\langle \mu\nu \rangle} + \rho_r^{\mu\nu} = 2\eta_r \sigma^{\mu\nu} + \text{h.o.t.} \tag{7}$$

- ▶ How to close this system?
  - → Power-counting scheme in Knudsen number $\mathrm{Kn} := \lambda_{\mathrm{mfp}}/\lambda_{\mathrm{hydro}}$ and inverse Reynolds numbers $\mathrm{Re}^{-1} := \delta f / f_0$ to second order

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$dK := d^3 k / [(2\pi)^3 k^0]$, $E_{\mathbf{k}} := u^\mu k_\mu$, matching conditions: $\rho_1 = \rho_2 = \rho_1^\mu = 0$

The DNMR approach

▶ General idea: **Diagonalize** the collision matrices
$\tau^{(\ell)} \equiv (\Omega^{(\ell)})^{-1} \text{diag}(\tau_1^{(\ell)}, \tau_2^{(\ell)}, \cdots) \Omega^{(\ell)}$, sort eigenvalues in decreasing order

  ■ *Separation of scales: The slowest microscopic modes are most important macroscopically*

▶ Relate irreducible moments to dissipative quantities, e.g.
$\rho_r^{\mu\nu} \simeq a_r \pi^{\mu\nu} + b_r \sigma^{\mu\nu}$

  $\rightarrow \dot{\rho}_r^{\langle\mu\nu\rangle} \sim \dot{\pi}^{\langle\mu\nu\rangle}, \dot{\sigma}^{\langle\mu\nu\rangle}$

▶ Discard terms of order $\mathcal{O}(\text{Kn}^2 \text{Re}^{-1})$ or higher

Shear-stress relaxation equation (DNMR)

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{K}^{\mu\nu} \tag{8}$$

▶ First-order contributions $\sim \mathcal{O}(\text{Re}^{-1})$ and $\sim \mathcal{O}(\text{Kn})$

▶ Second-order contributions $\sim \mathcal{O}(\text{Kn}\,\text{Re}^{-1})$ and $\sim \mathcal{O}(\text{Kn}^2)$

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▶ Consider the second-order terms of tensor-rank two:

$$
\begin{aligned}
\mathcal{J}^{\mu\nu} &= 2\tau_\pi \pi^{\langle\mu}_\lambda \omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \tau_{\pi n}n^{\langle\mu}F^{\nu\rangle} \\
&\quad + \ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} + \lambda_{\pi n}n^{\langle\mu}I^{\nu\rangle} , \tag{9} \\
\mathcal{K}^{\mu\nu} &= \tilde{\eta}_1 \omega^{\lambda\langle\mu}\omega^{\nu\rangle}_\lambda + \tilde{\eta}_2\theta\sigma^{\mu\nu} + \tilde{\eta}_3\sigma^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda + \tilde{\eta}_4\sigma^{\langle\mu}_\lambda\omega^{\nu\rangle\lambda} + \tilde{\eta}_5 I^{\langle\mu}I^{\nu\rangle} \\
&\quad + \tilde{\eta}_6 F^{\langle\mu}F^{\nu\rangle} + \tilde{\eta}_7 I^{\langle\mu}F^{\nu\rangle} + \tilde{\eta}_8\nabla^{\langle\mu}I^{\nu\rangle} + \tilde{\eta}_9\nabla^{\langle\mu}F^{\nu\rangle} \tag{10}
\end{aligned}
$$

▶ Second derivatives of fluid-dynamical quantities appear

→ Equations become **parabolic**!
→ Theory becomes acausal and thus unstable

▶ DNMR approach: **Ignore** terms of order $\mathcal{O}(\mathrm{Kn}^2)$

→ Equations are hyperbolic again

▶ Is there a way to ensure $\mathcal{K}^{\mu\nu} = 0$ from the beginning?

$F^\mu := \nabla^\mu P_0,\ \omega^{\mu\nu} := (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$

The IReD approach

▶ General idea: Relate moments through their Navier-Stokes solutions

$$\rho_r^{\mu\nu} = 2\eta_r \sigma^{\mu\nu} + \mathcal{O}(\mathrm{KnRe}^{-1})\,, \quad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \mathcal{O}(\mathrm{KnRe}^{-1})\,,$$

$$\Rightarrow \rho_r^{\mu\nu} = \frac{\eta_r}{\eta}\pi^{\mu\nu} + \mathcal{O}(\mathrm{KnRe}^{-1})\,. \tag{11}$$

$$\rightarrow \dot{\rho}_r^{\langle\mu\nu\rangle} \sim \dot{\pi}^{\langle\mu\nu\rangle}$$

▶ Discard terms of order $\mathcal{O}(\mathrm{Kn}^2\mathrm{Re}^{-1})$ or higher

Shear-stress relaxation equation (IReD)

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} \tag{12}$$

▶ Only terms $\sim \mathcal{O}(\mathrm{Re}^{-1})$, $\sim \mathcal{O}(\mathrm{Kn})$, $\sim \mathcal{O}(\mathrm{KnRe}^{-1})$ appear
  → Equations stay **hyperbolic**, no need to discard terms
  → Transport coefficients change
▶ How are the methods related?

Approach also known as "order-of-magnitude approach"

▶ Consider again the structure of the second-order terms of tensor-rank two:

$$
\begin{aligned}
\mathcal{J}_{\text{DNMR}}^{\mu\nu} &= \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + 2\tau_\pi\pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \lambda_{\pi n}n^{\langle\mu}I^{\nu\rangle} \\
&\quad -\tau_{\pi n}n^{\langle\mu}F^{\nu\rangle} + \ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} , \tag{13} \\
\mathcal{K}_{\text{DNMR}}^{\mu\nu} &= \tilde{\eta}_1\omega^{\lambda\langle\mu}\omega^{\nu\rangle}_\lambda + \tilde{\eta}_2\theta\sigma^{\mu\nu} + \tilde{\eta}_3\sigma^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \tilde{\eta}_4\sigma_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \tilde{\eta}_5I^{\langle\mu}I^{\nu\rangle} \\
&\quad +\tilde{\eta}_6F^{\langle\mu}F^{\nu\rangle} + \tilde{\eta}_7I^{\langle\mu}F^{\nu\rangle} + \tilde{\eta}_8\nabla^{\langle\mu}I^{\nu\rangle} + \tilde{\eta}_9\nabla^{\langle\mu}F^{\nu\rangle} . \tag{14}
\end{aligned}
$$

▶ Use Navier-Stokes solutions of moment equations to absorb terms in $\mathcal{K}_{\text{DNMR}}^{\mu\nu}$ into $\mathcal{J}_{\text{IReD}}^{\mu\nu}$ via $\theta \simeq -\Pi/\zeta$, $I^\mu \simeq n^\mu/\kappa$, $\sigma^{\mu\nu} \simeq \pi^{\mu\nu}/2\eta$

  ■ "Trade one power of Kn for one power of Re^{-1}"

▶ The terms in red can be related to
$\dot{\sigma}^{\langle\mu\nu\rangle} \sim -\omega^{\lambda\langle\mu}\omega^{\nu\rangle}_\lambda - \frac{\tilde{\eta}_6}{\tilde{\eta}_1}F^{\langle\mu}F^{\nu\rangle} - \frac{\tilde{\eta}_9}{\tilde{\eta}_1}\nabla^{\langle\mu}F^{\nu\rangle}$

▶ Since $\dot{\sigma}^{\langle\mu\nu\rangle} = \frac{1}{2\eta}\dot{\pi}^{\langle\mu\nu\rangle} - \frac{1}{2\eta^2}\pi^{\mu\nu}\dot{\eta}$, $\tilde{\eta}_1$ leads to a modification of τ_π:

$$
\tau_\pi^{\text{IReD}} = \tau_\pi^{\text{DNMR}} + \frac{\tilde{\eta}_1}{2\eta}. \tag{15}
$$

▶ Result: IReD and DNMR equivalent up to order $\mathcal{O}(\text{Kn}^2, \text{KnRe}^{-1}, \text{Re}^{-2})$
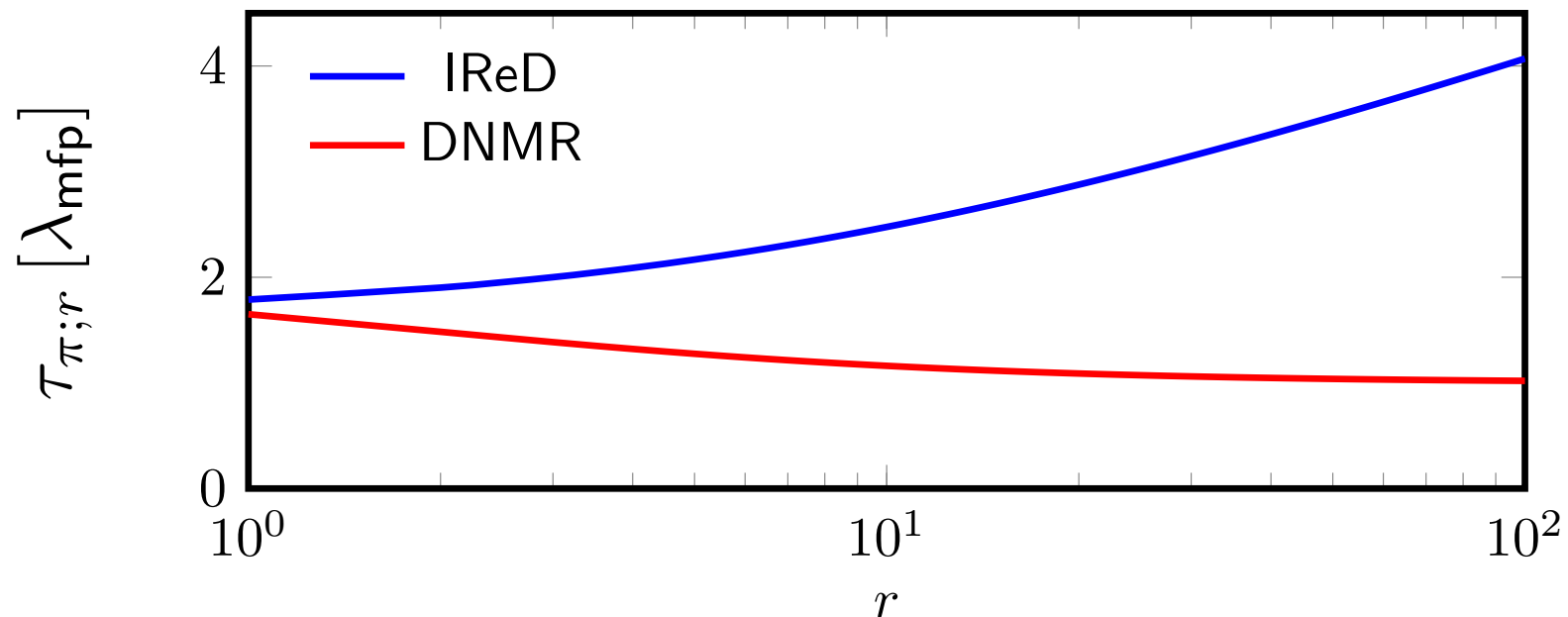
Test case: Ultrarelativistic hard spheres

▶ Simple model with constant cross-section: Linearized collision matrices can be calculated analytically

DW, V. E. Ambruș, E. Molnár, in preparation

IReD	Relation	DNMR
$\tau_\pi = 1.66\lambda_{\mathrm{mfp}}$	$\tau_\pi = \tilde{\tau}_\pi + \dfrac{\tilde{\eta}_1}{2\eta}$	$\tilde{\tau}_\pi = 2\lambda_{\mathrm{mfp}}$
$\tau_{\pi\pi} = 1.69\tau_\pi$	$\tau_{\pi\pi} = \tilde{\tau}_{\pi\pi} + \dfrac{\tilde{\eta}_1 - \tilde{\eta}_3}{2\eta}$	$\tilde{\tau}_{\pi\pi} = 1.69\tilde{\tau}_\pi$
$\ell_{\pi n} = -0.57\tau_\pi/\beta$	$\ell_{\pi n} = \tilde{\ell}_{\pi n} + \dfrac{\tilde{\eta}_8}{\kappa}$	$\tilde{\ell}_{\pi n} = -0.69\tilde{\tau}_\pi/\beta$

▶ Properly accounting for $\mathcal{K}^{\mu\nu}$ within IReD gives a 17% difference in τ_π, together with substantial differences in e.g. $\ell_{\pi n}/\tau_\pi$

▶ Fundamental idea of DNMR: Separation of microscopic timescales
- What happens to this idea in IReD?

▶ Consider relaxation equations for higher (non-hydrodynamic) moments
- DNMR: Relaxation times given by eigenvalues of inverse collision matrix
- IReD: Modification through absorption of $\mathcal{K}^{\mu\nu}$ into $\mathcal{J}^{\mu\nu}$

▶ Different behaviour in the two theories for $r \to \infty$:
- DNMR: $\tau_{\pi;r} \to \lambda_{\mathsf{mfp}}$
- IReD: $\tau_{\pi;r} \sim \log(r)$
  - → The *Separation of Scales* paradigm does not hold in IReD anymore!

Conclusion and Outlook

▶ The IReD approach to relativistic dissipative hydrodynamics relates irreducible moments ($\rho_r^{\mu\nu}$) directly to dissipative quantities ($\pi^{\mu\nu}$)

→ No terms $\sim \mathcal{O}(\mathrm{Kn}^2)$ appear in equations of motion

→ Equations stay **hyperbolic**, no modifications needed

▶ Relaxation times behave fundamentally different, *separation of scales* no longer valid

▶ IReD and DNMR are (perturbatively) equivalent

▶ **However**, in the regime where the $\mathcal{O}(\mathrm{Kn}^2)$ contributions are non-negligible, the IReD approach should do better

→ Future plan: Compare performance in different setups

Appendix

Hard spheres collision matrix

▶ The collision matrix is linked with the expansion of $\delta f_{\mathbf{k}}$ with respect to a complete basis,

$$\delta f_{\mathbf{k}} = f_{0\mathbf{k}} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \rho_n^{\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle} \mathcal{H}_{\mathbf{k}n}^{(\ell)},$$

where $\mathcal{H}_{\mathbf{k}n}^{(\ell)}$ is defined such that $\rho_n^{\mu_1 \cdots \mu_\ell} \equiv \int dK E_{\mathbf{k}}^n k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \delta f_{\mathbf{k}}$.

▶ The linearized collision integrals are given by

$$\mathcal{A}_{rn}^{(\ell)} = \frac{1}{\nu(2\ell+1)} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} E_{\mathbf{k}}^{r-1} k^{\langle \nu_1} \cdots k^{\nu_\ell \rangle}$$

$$\times \left(\mathcal{H}_{\mathbf{k}n}^{(\ell)} k_{\langle \nu_1} \cdots k_{\nu_\ell \rangle} + \mathcal{H}_{\mathbf{k}'n}^{(\ell)} k'_{\langle \nu_1} \cdots k'_{\nu_\ell \rangle} - \mathcal{H}_{\mathbf{p}n}^{(\ell)} p_{\langle \nu_1} \cdots p_{\nu_\ell \rangle} - \mathcal{H}_{\mathbf{p}'n}^{(\ell)} p'_{\langle \nu_1} \cdots p'_{\nu_\ell \rangle} \right),$$

▶ In the case of the UR ideal HS gas, $W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} = s(2\pi)^6 \delta^{(4)}(k + k' - p - p') \frac{\sigma_T \nu}{4\pi}$ and

$$\mathcal{A}_{r=0,n}^{(1)} = \frac{16(-\beta)^n g^2}{\lambda_{\mathrm{mfp}}(n+3)!} \left[S_n^{(1)}(N_1) - \frac{\delta_{n0}}{2} \right], \qquad \mathcal{A}_{r=0,n}^{(2)} = \frac{432 g^2 (-\beta)^n}{\lambda_{\mathrm{mfp}}(n+5)!} S_n^{(2)}(N_2),$$

$$\mathcal{A}_{r>0,n\leq r}^{(1)} = \frac{g^2 \beta^{n-r}(r+2)![n(r+4)-r]}{\lambda_{\mathrm{mfp}}(n+3)!r} \qquad \mathcal{A}_{r>0,n\leq r}^{(2)} = \frac{g^2 \beta^{n-r}(r+4)!(n+1)}{\lambda_{\mathrm{mfp}}(n+5)!r(r+1)}$$

$$\times \left(\delta_{nr} + \delta_{n0} - \frac{2}{r+1} \right), \qquad \times (9n + nr - 4r)\left(\delta_{nr} - \frac{2}{r+2} \right),$$

while $\mathcal{A}_{r>0,n>r}^{(1)} = \mathcal{A}_{r>0,n>r}^{(2)} = 0$ and $S_n^{(\ell)}(N_\ell) = \sum_{m=n}^{N_\ell} \binom{m}{n} \frac{1}{(m+\ell)(m+\ell+1)}$.