New exact solutions of non relativistic, viscous hydrodynamics

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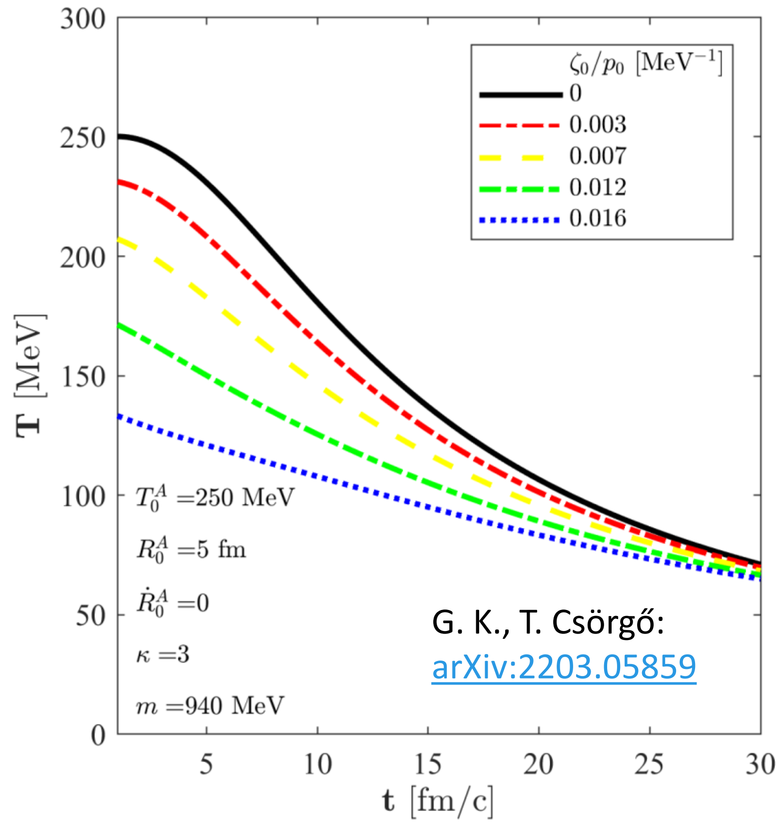
Introduction

Asymptotic perfection

- New exact solutions of viscous hydrodynamics have been found recently
- ***These solutions tends to perfect fluid solutions at asymptotically late times***
- **Perfect fluid attractor**: how to extract the effect of bulk and shear viscosity in final state measurments?



G. K., T. Csörgő: arXiv:2203.05859

Introduction

List of variety of viscous, asymptotically perfect fluid solutions:

Solutions were found…

- …in relativistic regime: T. Csörgő, G. K.: arXiv:2003.08859 M. Csanád, M. I. Nagy, Z. Jiang, T. Csörgő: arXiv:1909.02498
- …in non relativistic regime: G. K., T. Csörgő: arXiv:2203.05859 → ***Today's topic***

In the relativistic regime, we have found solutions of…

- …Israel-Stewart theory: M. Csanád, M. I. Nagy, Z. Jiang, T. Csörgő: arXiv:1909.02498 T. Csörgő, G. K.: arXiv:2003.08859
- …Navier-Stokes theory: M. Csanád, M. I. Nagy, Z. Jiang, T. Csörgő: arXiv:1909.02498 T. Csörgő, G. K.: arXiv:2003.08859 G. K., T. Csörgő: arXiv:2203.05859

Solutions were found with different assumptions, with different symmetries:

- $\zeta \sim p$, $\eta \sim p$ → spherical (only bulk), spheroidal, ellipsoidal T. Csörgő, G. K.: arXiv:2003.08859 G. K., T. Csörgő: arXiv:2203.05859
- $\zeta \sim n$ → spherical M. Csanád, M. I. Nagy, Z. Jiang, T. Csörgő: arXiv:1909.02498
- $\zeta = \zeta(p(t))$ → spherical, spheroidal, ellipsoidal (draft) G. K., T. Csörgő: arXiv:2203.05859 T. Csörgő, G. K.: arXiv:2003.08859

Introduction

Common properties of these solutions: Hubble-flow

Scales of the fireball: $X(t),\ Y(t),\ Z(t)$

Transverse scale: $R(t)$

Spherical symmetry: $\vec{v}=\frac{\dot{R}}{R}(r_x,r_y,r_z)$

Spheroidal symmetry: $v_H(\vec{r},t)=\left(\frac{\dot{R}}{R}r_x,\frac{\dot{Y}}{Y}r_y,\frac{\dot{R}}{R}r_z\right),\quad v_{rot}(\vec{r},t)=\omega(r_z,0,-r_x)\ ,\quad \dot{\vartheta}=\omega(t)$

Ellipsoidal symmetry:

$$v_H(\vec{r},t)=\begin{pmatrix}\left(\frac{\dot{X}}{X}\cos^2\vartheta+\frac{\dot{Z}}{Z}\sin^2\vartheta\right)r_x\\ \frac{\dot{Y}}{Y}r_y\\ \left(\frac{\dot{X}}{X}\sin^2\vartheta+\frac{\dot{Z}}{Z}\cos^2\vartheta\right)r_z\end{pmatrix}+\left(\frac{\dot{Z}}{Z}-\frac{\dot{X}}{X}\right)\frac{\sin(2\vartheta)}{2}\begin{pmatrix}r_z\\0\\r_x\end{pmatrix}$$

$$v_{rot}(\vec{r},t)=\dot{\vartheta}\begin{pmatrix}r_z\\0\\-r_x\end{pmatrix}+\dot{\vartheta}\begin{pmatrix}(\frac{X}{Z}\cos^2\vartheta+\frac{Z}{X}\sin^2\vartheta)r_z\\0\\-(\frac{X}{Z}\sin^2\vartheta+\frac{Z}{X}\cos^2\vartheta)r_x\end{pmatrix}+\dot{\vartheta}\left(\frac{X}{Z}-\frac{Z}{X}\right)\frac{\sin(2\vartheta)}{2}\begin{pmatrix}r_x\\0\\-r_z\end{pmatrix},\quad \dot{\vartheta}=\frac{\omega(t)}{2}=\frac{\omega_0}{2}\frac{R_0^2}{R(t)^2}$$

M. I. Nagy, T. Csörgő: arXiv:1309.4390
M. I. Nagy, T. Csörgő: arXiv:1606.09160
T. Csörgő, M. I. Nagy, I. F. Barna: arXiv:1511.02593

Introduction

Common properties of these solutions: Self-similarity $(\partial_t + \vec{v}\nabla)s = 0$

Spherical symmetry: $s = \frac{r^2}{R^2}$

Spheroidal symmetry: $s = \frac{r_x^2 + r_z^2}{R^2} + \frac{r_y^2}{Y^2}$

Ellipsoidal symmetry: $s = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2} + \left(\frac{1}{Z^2} - \frac{1}{X^2}\right)\left[(r_x^2 - r_z^2)\sin^2\vartheta + r_x r_z \sin(2\vartheta)\right]$

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Introduction

Non relativistic hydrodynamics

- The same effects can be understood in a much simpler formalism
- ***The basic equations of non relativistic, viscous hydro are fully clarified***

Local conservation laws:

$$\partial_t n + \nabla(n\vec{v}) = 0$$

$$\partial_t \varepsilon + \nabla(\varepsilon \vec{v}) + p\nabla\vec{v} = \zeta(\nabla\vec{v})^2 + 2\eta\left[\mathrm{Tr}\left(D^2\right) - \frac{1}{3}(\nabla\vec{v})^2\right]$$

$$(\varepsilon + p)(\partial_t + \vec{v}\nabla)\vec{v} + \nabla p = \nabla(\zeta\nabla\vec{v}) + \eta\left[\Delta\vec{v} + \frac{1}{3}\nabla(\nabla\vec{v})\right]$$

Balance eq. of entropy:

$$\partial_t \sigma + \nabla(\sigma\vec{v}) = \frac{\zeta}{T}(\nabla\vec{v})^2 + \frac{2\eta}{T}\left[\mathrm{Tr}\left(D^2\right) - \frac{1}{3}(\nabla\vec{v})^2\right] \geq 0$$

Introduction

To close the equation system:

EoS: $\varepsilon = \kappa(T) p$

$p = nT$

G. K., T. Csörgő: arXiv:2203.05859

Relationship to the speed of sound

κ is constant:

$$c_s^2(T) = \left(1 + \kappa^{-1}\right) \frac{T}{m}$$

κ is temperature dependent:

$$c_s^2(T) = \left[1 + \left(\kappa + T\frac{d\kappa}{dT}\right)^{-1}\right] \frac{T}{m}$$

γ(T): temperature dependent adiabatic index

New solutions

Self-similar solutions: product of a homogeneous term and arbitrary functions of *s*

$$n(t,s) = n_H(t)\mathcal{V}(s) \longrightarrow n(t,s) = n_0\frac{V_0}{V}\mathcal{V}(s)$$

$$T(t,s) = T_H(t)\mathcal{T}(s) \longrightarrow$$

$$p(t,s) = n(t,s)T(t,s) = p_H(t)\mathcal{V}(s)\mathcal{T}(s)$$

For constant κ we assume:

$$T_H(t) = T_P(t)T_D(t)$$

ν(s) and τ(s) are not independent from each other:

$$-\frac{C_E}{2} = \mathcal{T}'(s) + \frac{\mathcal{T}(s)}{\mathcal{V}(s)}\mathcal{V}'(s)$$

Parametric solutions with inhomogeneous pressure
(Spherical symmetry)

If the pressure is inhomogeneous, then ζ has to be linear in pressure: $\zeta(t,s) = \zeta_0 \frac{p(t,s)}{p_0}$

Assumption:

$$T_H(t) = T_D(t)\left(\frac{R_0}{R}\right)^{\frac{d}{\kappa}}$$

With that, the energy conservation becomes:

$$\frac{\dot{T}_D}{T_D} = \underbrace{\frac{\zeta_0 d^2}{\kappa p_0}\left(\frac{\dot{R}}{R}\right)^2}_{\text{effect of bulk viscosity}}$$

The Euler equation is:

$$R\ddot{R} = C_E \frac{T_H}{m}\left(1 - \frac{\zeta_0}{p_0}\underbrace{\frac{d\dot{R}}{R}}_{\text{effect of bulk viscosity}}\right)$$

Parametric solutions with inhomogeneous pressure
(Spheroidal symmetry)

If the pressure is inhomogeneous, then ζ and η has to be linear in pressure: $\frac{\zeta(t,s)}{\zeta_0} = \frac{\eta(t,s)}{\eta_0} = \frac{p(t,s)}{p_0}$

Assumption:

$$T_H(t) = T_D(t)\left(\frac{R_0^2 Y_0}{R^2 Y}\right)^{\frac{1}{\kappa}}$$

With that, the energy conservation becomes:

$$\frac{\dot{T}_D}{T_D} = \underbrace{\frac{\zeta_0}{\kappa p_0}\left(\frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y}\right)^2}_{\text{effect of bulk viscosity}} + \underbrace{\frac{4\eta_0}{3\kappa p_0}\left(\frac{\dot{R}}{R} - \frac{\dot{Y}}{Y}\right)^2}_{\text{effect of shear viscosity}}$$

The Euler equation is:

$$R\ddot{R} = Y\ddot{Y} = C_E \frac{T_P T_D}{m}\left[1 - \underbrace{\frac{\zeta_0}{p_0}\left(\frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y}\right)}_{\text{effect of bulk viscosity}}\right]$$

Parametric solutions with inhomogeneous pressure
(Spheroidal symmetry, rotation)

If the pressure is inhomogeneous, then ζ and η has to be linear in pressure: $$\frac{\zeta(t,s)}{\zeta_0} = \frac{\eta(t,s)}{\eta_0} = \frac{p(t,s)}{p_0}$$

Assumption:

$$T_H(t) = T_D(t)\left(\frac{R_0^2 Y_0}{R^2 Y}\right)^{\frac{1}{\kappa}}$$

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The Euler equation is:

$$R\ddot{R} - \underbrace{R^2\omega^2}_{\text{effect of rotation}} = Y\ddot{Y} = C_E \frac{T_P T_D}{m}\left[1 - \underbrace{\frac{\zeta_0}{p_0}\left(\frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y}\right)}_{\text{effect of bulk viscosity}}\right]$$

Parametric solutions with inhomogeneous pressure
(Ellipsoidal symmetry, rotation)

Pressure is inhomogeneous → ζ and η are linear in pressure: $\frac{\zeta(t,s)}{\zeta_0} = \frac{\eta(t,s)}{\eta_0} = \frac{p(t,s)}{p_0}$

Assumption:

$$T_H(t) = T_D(t)\left(\frac{X_0 Y_0 Z_0}{XYZ}\right)^{\frac{1}{\kappa}}$$

With that, the energy conservation becomes:

$$\frac{\dot{T}_D}{T_D} = \underbrace{\frac{\zeta_0}{\kappa p_0}\left(\frac{\dot{X}}{X}+\frac{\dot{Y}}{Y}+\frac{\dot{Z}}{Z}\right)^2}_{\text{effect of bulk viscosity}} + \overbrace{\underbrace{\frac{2\eta_0}{\kappa p_0}\left[\frac{\dot{X}^2}{X^2}+\frac{\dot{Y}^2}{Y^2}+\frac{\dot{Z}^2}{Z^2}-\frac{1}{3}\left(\frac{\dot{X}}{X}+\frac{\dot{Y}}{Y}+\frac{\dot{Z}}{Z}\right)^2\right]}_{\text{the only effect of shear viscosity for spheroidally symmetric fireball}} + \underbrace{\frac{\eta_0 {\omega_0}^2}{4\kappa p_0}\frac{(X_0+Z_0)^4}{(X+Z)^4}\left(\frac{X}{Z}-\frac{Z}{X}\right)^2}_{\text{effect of rotation, if } \eta\neq 0}}^{\text{effect of shear viscosity for ellipsoidally symmetric fireball}}$$

The Euler equation is (where $R=(X+Z)/2$):

$$X\big(\ddot{X}\underbrace{-R\omega^2}_{\text{effect of rotation}}\big) = Y\ddot{Y} = Z\big(\ddot{Z}\underbrace{-R\omega^2}_{\text{effect of rotation}}\big) = C_E\frac{T_P T_D}{m}\left[1\underbrace{-\frac{\zeta_0}{p_0}\left(\frac{\dot{X}}{X}+\frac{\dot{Y}}{Y}+\frac{\dot{Z}}{Z}\right)}_{\text{effect of bulk viscosity}}\right]$$

Parametric solutions with inhomogeneous pressure
(Ellipsoidal symmetry, rotation and *T* dependent *κ*)

Pressure is inhomogeneous → ζ and η are linear in pressure: $\frac{\zeta(t,s)}{\zeta_0} = \frac{\eta(t,s)}{\eta_0} = \frac{p(t,s)}{p_0}$

No assumption for $f_T(t)$ and we define the scale volume:

$$T(t,s) = T_H(t)\mathcal{T}(s) \qquad T_H(t) \neq T_P(t)T_D(t) \qquad V \equiv V(t) = (2\pi)^{3/2}XYZ$$

The energy conservation becomes:

$$\left(\frac{1}{\gamma(T)-1}\right)\frac{\dot{T}_D}{T_D} + \frac{\dot{V}}{V} = \frac{\zeta_0}{p_0}\left(\frac{\dot{V}}{V}\right)^2 + \frac{2\eta_0}{p_0}\left[\frac{\dot{X}^2}{X^2} + \frac{\dot{Y}^2}{Y^2} + \frac{\dot{Z}^2}{Z^2} - \frac{1}{3}\left(\frac{\dot{V}}{V}\right)^2\right] + \frac{\eta_0\omega_0{}^2}{4p_0}\frac{(X_0+Z_0)^4}{(X+Z)^4}\left(\frac{X}{Z} - \frac{Z}{X}\right)^2$$

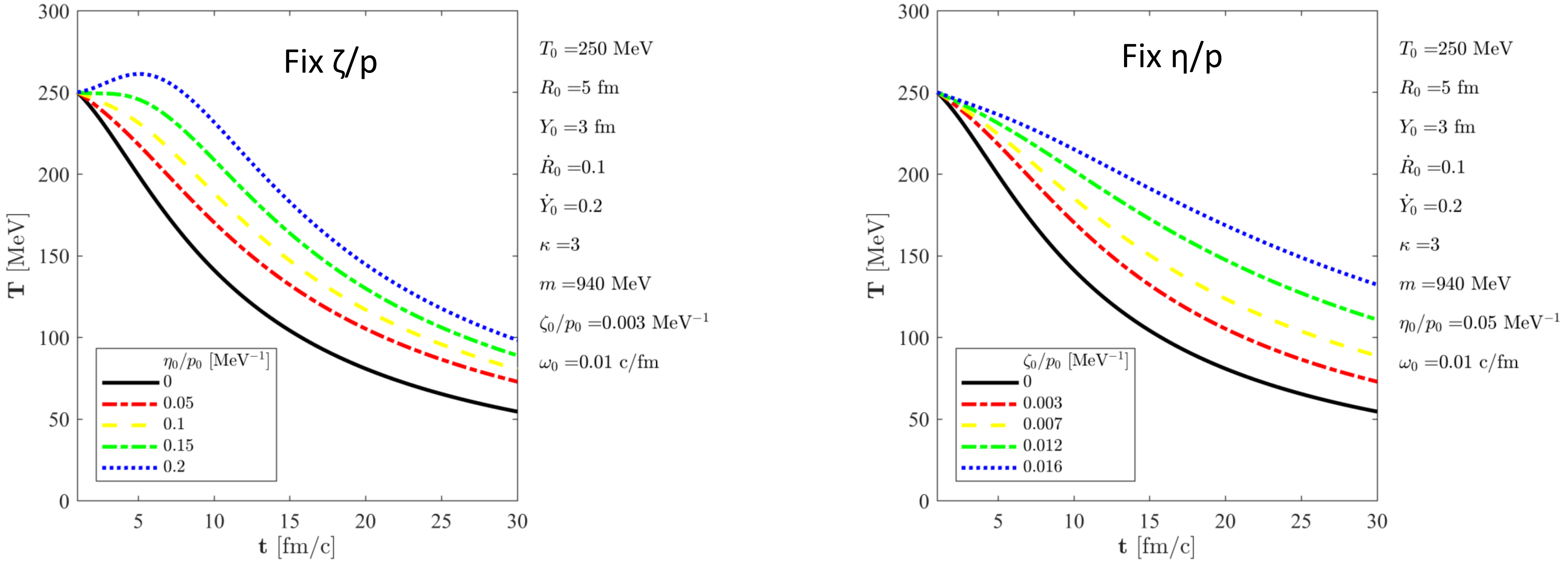
- effect of bulk viscosity: $\frac{\zeta_0}{p_0}\left(\frac{\dot{V}}{V}\right)^2$
- the only effect of shear viscosity for spheroidally symmetric fireball: $\frac{2\eta_0}{p_0}\left[\dots\right]$
- effect of rotation, if η≠0: $\frac{\eta_0\omega_0{}^2}{4p_0}\frac{(X_0+Z_0)^4}{(X+Z)^4}\left(\frac{X}{Z} - \frac{Z}{X}\right)^2$
- effect of shear viscosity for ellipsoidally symmetric fireball: the last two terms together

The Euler equation:

$$X\left(\ddot{X} - R\omega^2\right) = Y\ddot{Y} = Z\left(\ddot{Z} - R\omega^2\right) = C_E\frac{T_H}{m}\left[1 - \frac{\zeta_0}{p_0}\left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right)\right]$$

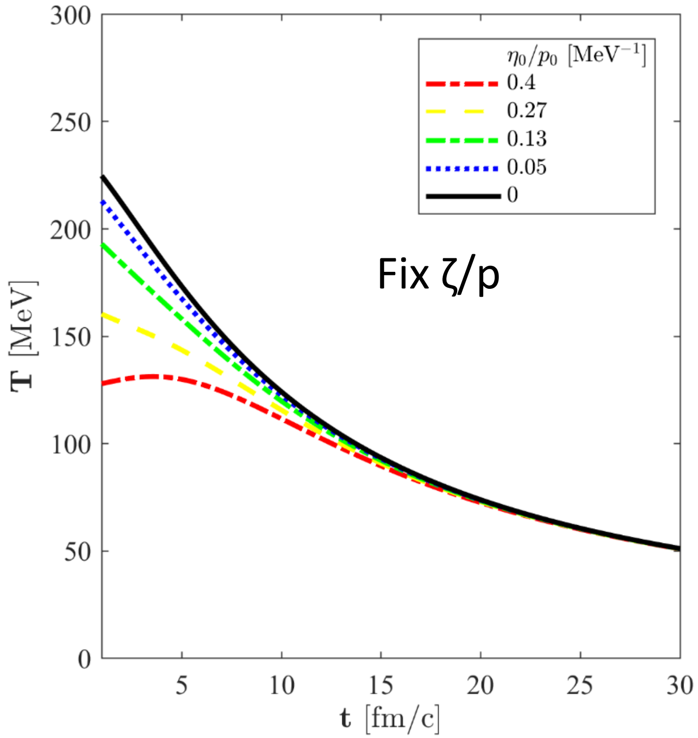
- effect of rotation: the $-R\omega^2$ terms (in the X and in the Z equation)
- effect of bulk viscosity: $\frac{\zeta_0}{p_0}\left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right)$

Spheroidally symmetric, parametric solution: Evolution of temperature



Fixed initial conditions

Spheroidally symmetric, parametric solution: Evolution of temperature



$T_{A,0}$ =225 MeV

$R_{A,0}$ =5.53 fm

$\dot{R}_{A,0}$ =0.25

$Y_{A,0}$ =3.38 fm

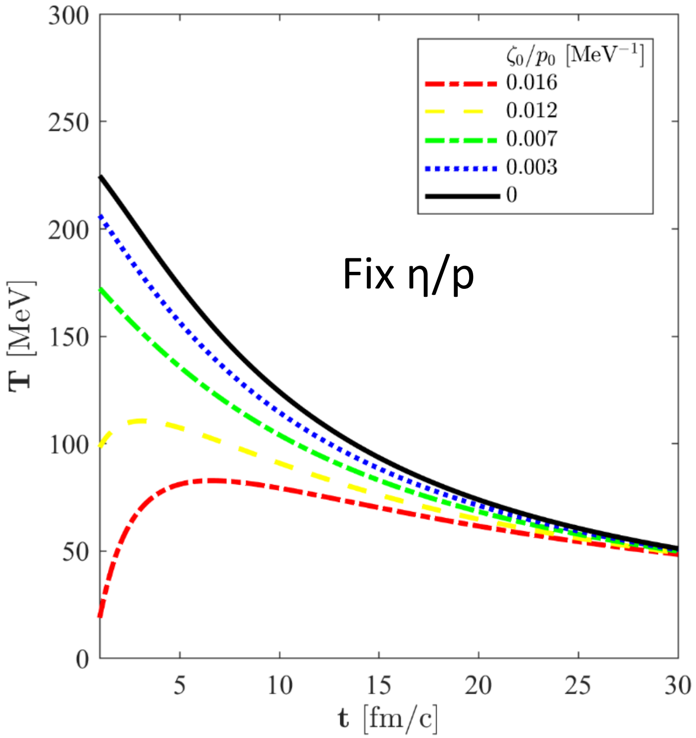
$\dot{Y}_{A,0}$ =0.24

κ =3

m =940 MeV

ζ_0/p_0 =0 MeV^{-1}

$\omega_{A,0}$ =0.01 c/fm



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κ =3

m =940 MeV

η_0/p_0 =0 MeV^{-1}

$\omega_{A,0}$ =0.01 c/fm

Fixed attractor

Spheroidally symmetric, analytic solution

The pressure is homogeneous

There is no rotation in the system

$$R(t) = R_0 + \dot{R}_0(t - t_0),$$

$$Y(t) = Y_0 + \dot{Y}_0(t - t_0),$$

The solution of particle density is same as before

We use: $T_H(t) = T_P(t) T_D(t)$

The temperature equation is:

$$\kappa_0 \partial_t \ln(T_D) = \frac{\xi_R}{(t + \Delta t_R)^2} + \frac{\xi_Y}{(t + \Delta t_Y)^2} + \frac{\xi_{RY}}{(t + \Delta t_R)(t + \Delta t_Y)}$$

New time-like parameters:

$$\Delta t_R = H_R^{-1} - t_0 = \frac{R_0}{\dot{R}_0} - t_0$$

$$\Delta t_Y = H_Y^{-1} - t_0 = \frac{Y_0}{\dot{Y}_0} - t_0$$

Assumptions for viscosities:

$$\frac{\zeta(t,s)}{\zeta_0} = \frac{\eta(t,s)}{\eta_0} = \frac{p(t,s)}{p_0}$$

The viscosities are carried by:

$$\xi_R = \frac{4\zeta_0}{p_0} + \frac{4\eta_0}{3p_0}$$

$$\xi_Y = \frac{\zeta_0}{p_0} + \frac{4\eta_0}{3p_0}$$

$$\xi_{RY} = \frac{4\zeta_0}{p_0} - \frac{8\eta_0}{3p_0}$$

Spheroidally symmetric, analytic solution

The solution of the dissipative correction is:

$$T_D(t) = \left(\frac{t_0+\Delta t_R}{t+\Delta t_R}\right)^{\frac{\xi_{RY}}{\kappa_0\left(H_R^{-1}-H_Y^{-1}\right)}} \left(\frac{t_0+\Delta t_Y}{t+\Delta t_Y}\right)^{\frac{\xi_{RY}}{\kappa_0\left(H_Y^{-1}-H_R^{-1}\right)}} \exp\left[\frac{\xi_R H_R}{\kappa_0}\left(1-\frac{t_0+\Delta t_R}{t+\Delta t_R}\right)+\frac{\xi_Y H_Y}{\kappa_0}\left(1-\frac{t_0+\Delta t_Y}{t+\Delta t_Y}\right)\right]$$

For asymptotically late times the dissipative correction is constant:

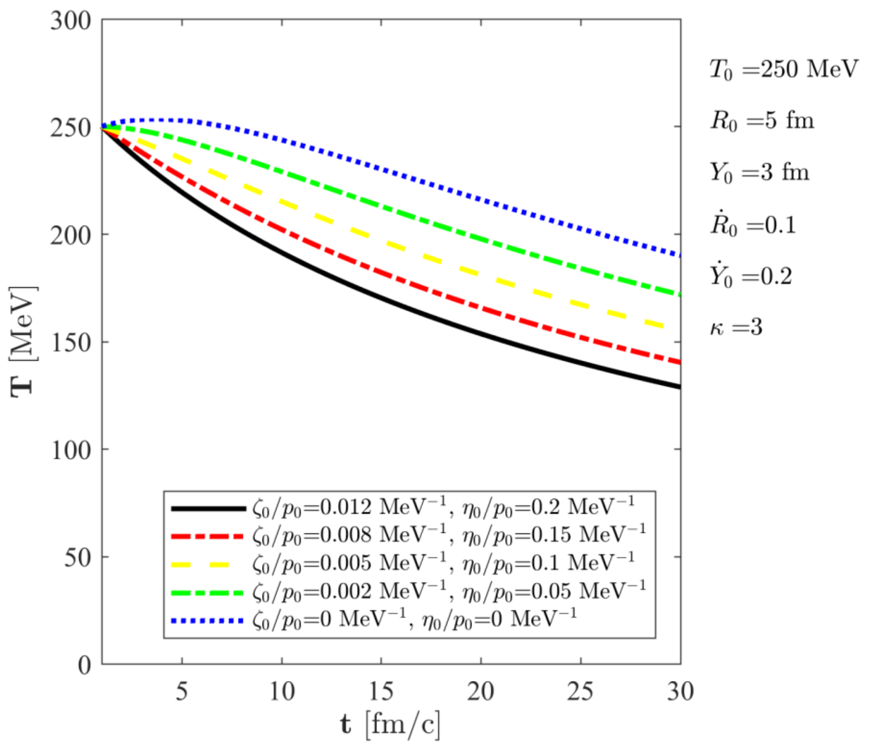
$$T_D(\kappa_0) = \left(\frac{t_0+\Delta t_R}{t_0+\Delta t_Y}\right)^{\frac{\xi_{RY}}{\kappa_0\left(H_R^{-1}-H_Y^{-1}\right)}} \exp\left(\frac{\xi_R H_R}{\kappa_0}+\frac{\xi_Y H_Y}{\kappa_0}\right)$$

For late time, T_D does not affect the time evolution, but rescale the initial temperature

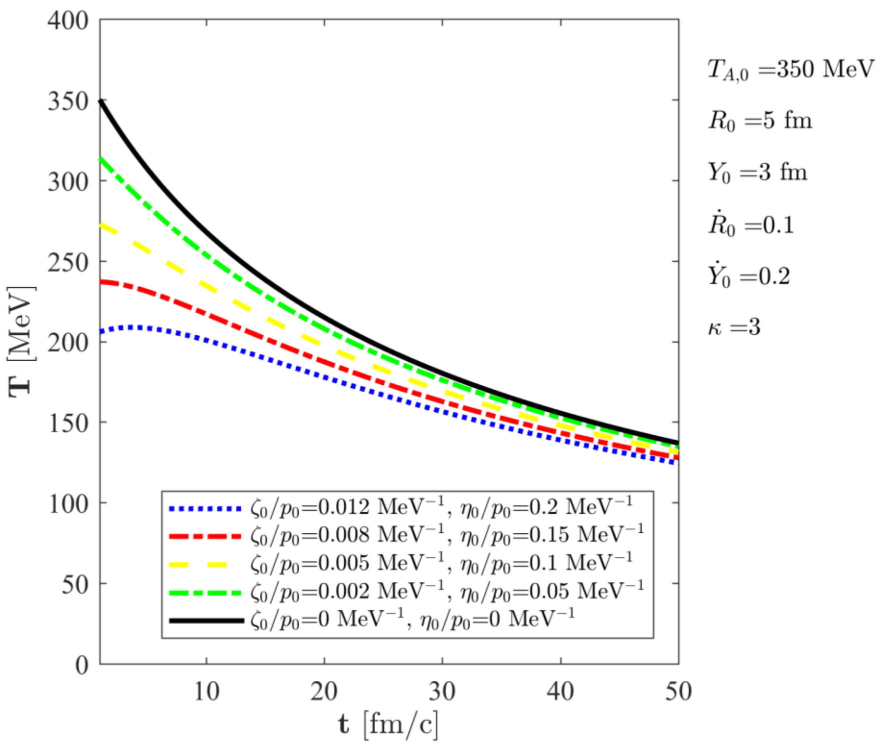
This analytic solution tends to a perfect fluid solution:

$$T_A(t,s) = T_{A,0}(\kappa_0)\left(\frac{R_0^2 Y_0}{R^2 Y}\right)^{\frac{1}{\kappa_0}} \mathcal{T}(s)$$

Spheroidally symmetric, analytic solution



Fixed initial conditions



Fixed attractor

Spheroidally symmetric, analytic solution

Analytic solution with T dependent κ

Temperature equation:

$$\left[\frac{d}{dT}(\kappa T)\right]\partial_t \ln(T_H) + \frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y} = \frac{\zeta_0}{p_0}\left(\frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y}\right)^2 + \frac{4\eta_0}{3p_0}\left(\frac{\dot{R}}{R} - \frac{\dot{Y}}{Y}\right)^2$$

We have seen that it can be solved analytically, if κ is constant

There is another possibility:

$$\frac{d}{dT}[\kappa(T)T] = \Delta\kappa_T = \text{constant}$$

$$\Delta\kappa_T\, \partial_t \ln(T_D) = \frac{\xi_R}{(t+\Delta t_R)^2} + \frac{\xi_Y}{(t+\Delta t_Y)^2} + \frac{\xi_{RY}}{(t+\Delta t_R)(t+\Delta t_Y)}$$

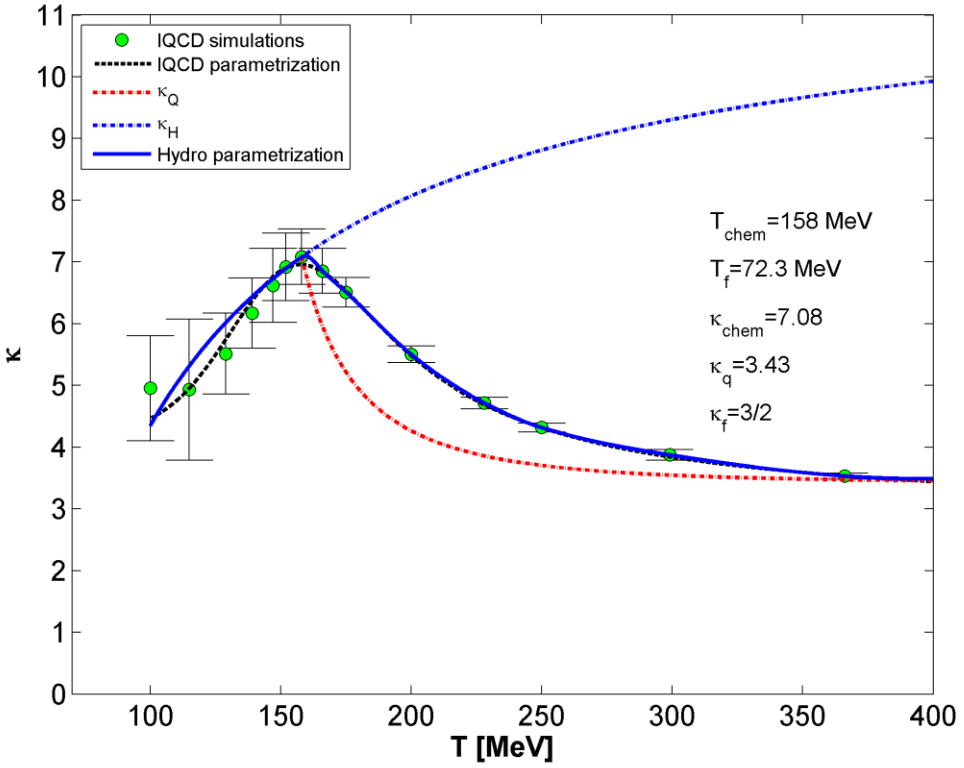
The same temperature eq. as before with different constant coefficient:
$\kappa_0 \rightarrow \Delta\kappa_T$

Spheroidally symmetric, analytic solution

Temperature dependent function for κ is obtained earlier → lattice QCD EoS was parametrized

arXiv:1610.02197 and my MSc thesis

We have provided an analytic, viscous solution with lattice QCD EoS!



Spheroidally symmetric, analytic solution

One more interesting property…

For fixed attractor and kinematic shear viscosity:

→ *the kinematic bulk viscosity is a non monotonic function of the initial temperature*

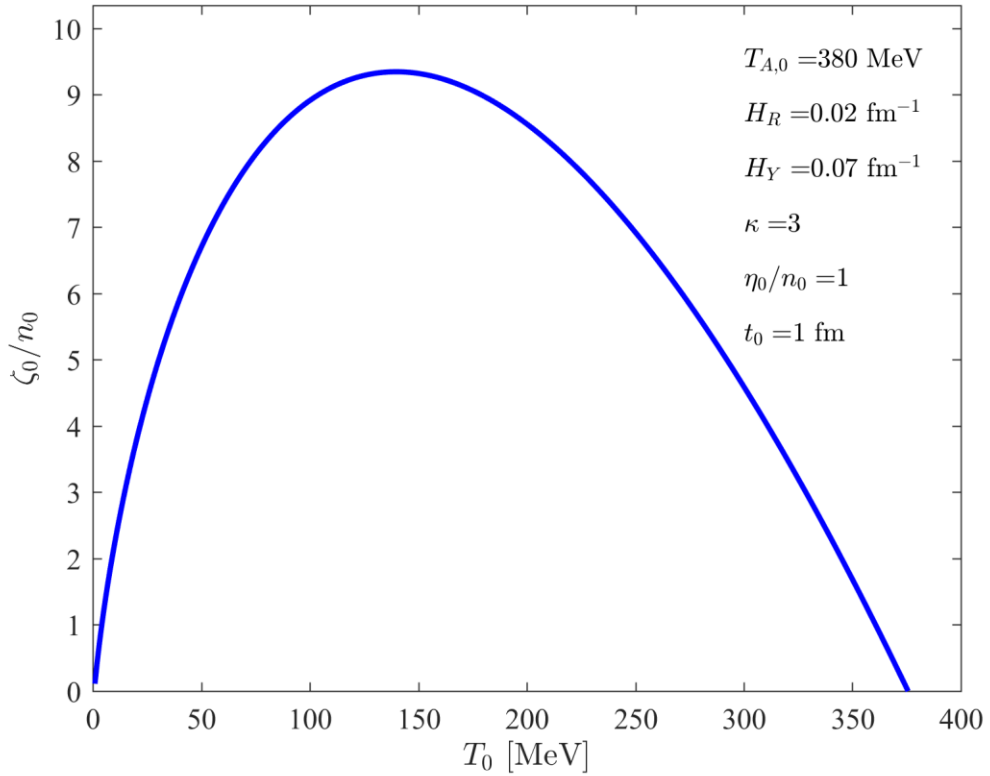
For fixed attractor and kinematic bulk viscosity:

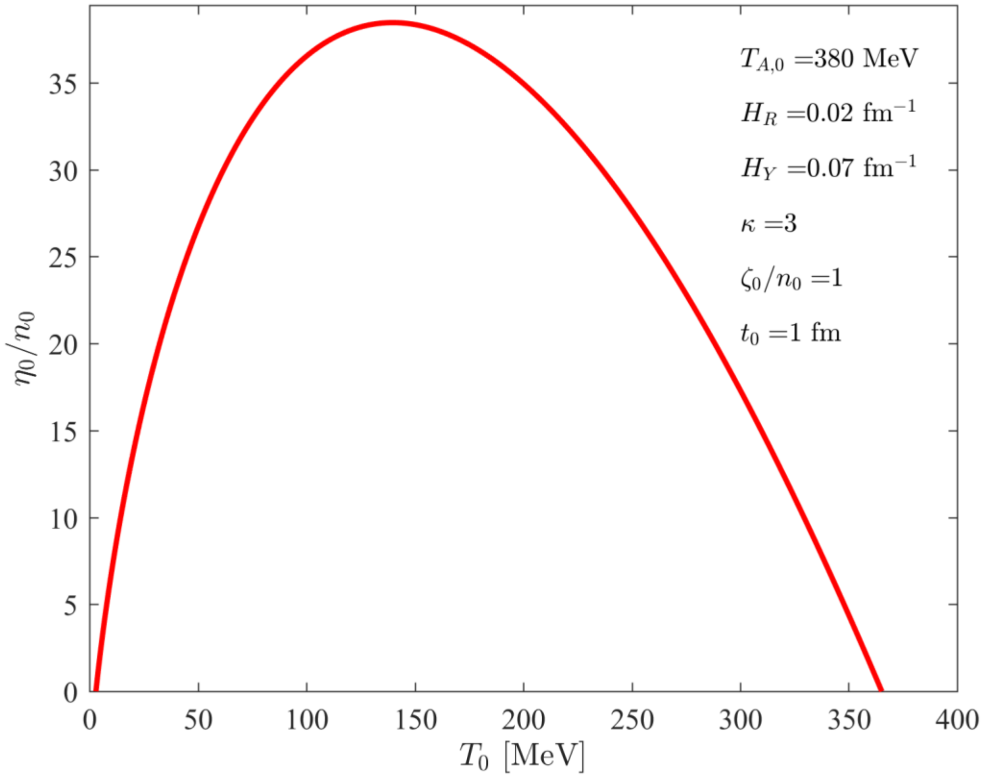
→ *the kinematic shear viscosity is a non monotonic funcion of the initial temperature*

$$\kappa_0 T_0 \ln\left(\frac{T_{A,0}(\kappa_0)}{T_0}\right) = \frac{\zeta_0}{n_0}\left[4H_R + H_Y - \frac{4H_R H_Y}{H_R+H_Y}\ln\left(\frac{t_0+\Delta t_Y}{t_0+\Delta t_R}\right)\right] + \frac{4\eta_0}{3n_0}\left[H_R + H_Y + \frac{2H_R H_Y}{H_R+H_Y}\ln\left(\frac{t_0+\Delta t_Y}{t_0+\Delta t_R}\right)\right]$$

Spheroidally symmetric, analytic solution

Non monotonic shear and bulk viscosity as functions of the initial temperature





Summary

New, parametric and analytic solutions of non relativistic hydrodynamics have been found

Common property of these solutions: **Hubble-flow**

Only academic results, *not plan to describe measurements*

The parametric solutions with inhomogeneous pressure are fully developed

- *(ellipsoidal symmetry, rotation is included, temperature dependent κ is allowed, non-constant ζ and η)*

Recent result: Spheroidally symmetric, analytic solutions with homogeneous pressure

- *(temperature dependent κ is allowed with a certain condition, non-constant ζ and η)*

All of the presented solutions are asymptotically perfect and tend to perfect fluid solutions

The analytic solution has indicated **non monotonic behaviour of the kinematic viscosities**

Thank you for your attention!