


Mechanisms behind the Tsallis-Pareto statistics

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Mathematical mechanisms

an incomplete overview

- a) Altered entropy formula (Rényi, Chravda-Hrvat, Tsallis, Thurner, Biró, . . .)
- b) Altered energy constraint **only** for special interaction classes *EPL 84, 56003, 2008*
- c) Non-linear master eq. (generalize: Fokker-Planck, Boltzmann eq., H-theorem) *PRL 95, 162302, 2005*
- d) Energy dependent noise → Biró at.al. QM 2005 *PRL 94, 132302, 2005*
- e) LGGR: linear growth rate, constant reset rate Biró, Nédá, et.al. 2019 *PhysA 499, 335, 2018*



Motivation

for a) entropy formulas

- math phantasy (need for generalization)
- purposeful design (e.g. to get power-law)
- finite heat bath effect (calculable for ideal gas) *PhysA 392, 3132, 2013*
- deformed addition (composition algebra, formal logarithm) *JPG 37, 094027, 2010*



Motivation

for b) nonadditive energy composition

$$E_{12} = \epsilon_1 V_1 + \epsilon_2 V_2 + G_{12}(V_1 \cap V_2) \quad V_1 \cap V_2 = A_{12} \ell_{\text{int}}$$

- long range interaction
- fractal interface
- edge of chaos (weak chaos)
- ultrarelativistic, int depends on

$$Q^2 = 2E_1 E_2 (1 - \cos \Theta_{12})$$

$$E_{12} = E_1 + E_2 + G(E_1 \cdot E_2).$$



Insert: Abstract composition rules

formal logarithm

$$x \oplus y = h(x, y) \quad (1)$$

with $h(x, 0) = x$, $h(y, 0) = y$ and associativity $h(h(x, y), z) = h(x, h(y, z))$.

The formal logarithm, $K(x \oplus y) = K(x) + K(y)$, is a map to addition.

Its partial derivative against y at $y = 0$:

$$K'(h(x, 0)) \left. \frac{\partial}{\partial y} h(x, y) \right|_{y=0} = K'(0) \quad (2)$$

reveals how to obtain form.log. from the rule:

$$K(x) = \int_0^x \frac{du}{\left. \frac{\partial}{\partial v} h(u, v) \right|_{v=0}} \quad (3)$$

Composition rules

examples for form.log.-s

Original rule: $h(x, y)$, asymptotic attractor: $i(x, y) = K^{-1}(K(x) + K(y))$.

● $h(x, y) = x + y \rightarrow K(x) = x$, attractor: $i(x, y) = x + y$

● $h(x, y) = x + y + G(xy) \rightarrow K(x) = \frac{1}{G'(0)} \ln(1 + G'(0)x)$,

and $i(x, y) = x + y + G'(0)xy$.

● $h(u, v) = \frac{u+v}{1+uv/c^2} \rightarrow K(u) = c \operatorname{atanh}(u/c)$,

and $i(u, v) = h(u, v)$.

For stable rules $i(x, y) = h(x, y)$

Non-Extensivity $q \neq 1$

size dependence of non-additive parameter

Observe/assume: $S_{12} = S_1 + S_2 + (q - 1)S_1 S_2.$

multiply, $X = (q - 1)S,$

conclude $X_{12} = X_1 + X_2 + X_1 X_2.$

The X-rule is universal: size independent, $\text{sizeof}(X) \sim \mathcal{O}(1).$

1. $\text{sizeof}(S) \sim \mathcal{O}(N) \rightarrow q - 1 \sim \mathcal{O}(1/N)$

2. $q - 1 \gg \mathcal{O}(1/N) \rightarrow \text{sizeof}(S) \ll \mathcal{O}(N).$



Motivation

for c) nonlinear stochasticity

- econophysics models *PhysA 387, 1603, 2008*
- stoichiometric factors in physical chemistry (3-gluon processes)



Motivation

for d) energy dependent noise

$$\dot{p}_i + \Gamma_{ij}(E)p_j = \zeta_i, \quad \langle \zeta_i \zeta_j \rangle = 2D_{ij}(E)\delta(t - t'), \quad D_{ij}(E) = T \left(\Gamma_{ij}(E) + D'_{ij}(E) \right)$$

- non-Abelian plasma: colored noise *PRL 94, 132302, 2005*
- insurance models: risk calculation
- quality control in industry
- hadronization *EPJA 40, 325, 2009*

The total energy dependent individual noise calls for finite reservoirs



Motivation

for e) LGGR reset dynamics

- Evolution: still vs catastrophic periods
- Income and wealth distribution
- Distribution of citations, likes and shares
- A certain hadronization model for QGP with re-heating

PhysA 499, 335, 2018

$$e^{-\frac{E}{T}} \rightarrow e^{-\int \frac{dE}{T(E)}} = \frac{1}{D(E)} e^{-\int \frac{\Gamma(E)}{D(E)} dE}. \quad (4)$$



Below the thermodynamical limit

Ideal gas = constant heat capacity

Boltzmann: $P = k \log W$.

Boltzmann: $P \propto 1/\Omega$.

Einstein: $\Omega = e^{S(E)}$

Sickur–Tetrode: $\Omega(E) \propto E^N$

Avogadro: $6 \cdot 10^{23}$, $1/\sqrt{N} \approx 10^{-10}$;

neurons in human brain: 10^{11} , $1/\sqrt{N} \approx 0,03\%$;

new particles in HIC: 6000, $1/\sqrt{N} \approx 1\%$;

multiplicity in pp: 6 - 60, $1/\sqrt{N} \approx 40\%$.



One particle energy distribution

from phase space volume ratio

$$P_1(\omega) = \frac{\Omega_1(\omega)\Omega_n(E-\omega)}{\Omega_{n+1}(E)} = w_1(\omega) \cdot \frac{(E-\omega)^n}{E^n} \quad (5)$$

n may fluctuate.

Let the PDF be P_n . Then the effective 1-PTL energy distribution is

$$P_1^{\text{eff}}(\omega) = \sum_{n=0}^{\infty} P_n \left(1 - \frac{\omega}{E}\right)^n. \quad (6)$$



Phase space dimension fluctuation

Blind chance models

distribute n particles among k cells: for repeated combination (bosons) we get $\binom{n+k}{n}$ possibilities.

$$\binom{n+k}{k}$$

Blind chance subspace:

$$B_{n,k}(f) \equiv \lim_{K \rightarrow \infty} \frac{\binom{n+k}{n} \binom{N-n+K-k}{N-n}}{\binom{N+K+1}{N}} = \binom{n+k}{n} f^n (1+f)^{-n-k-1}. \quad (7)$$

Here $f = N/K$ kept fixed.

There are other mechanisms resulting NBD

Phase space dim according to NBD

interpreting T and q

E fixed, N fluctuates

$$P_1^{\text{eff}}(\omega) = \sum_{N=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^N B_{N,K}(f) = \left(1 + f \frac{\omega}{E}\right)^{-K-1} \quad (8)$$

Here $\langle N \rangle = f(K + 1)$. Compare with TP-distribution and gain

$$T = \frac{E}{\langle N \rangle}, \quad q = 1 + \frac{1}{K + 1}. \quad (9)$$

Thermodynamical limit

Boltzmann, Poisson

The limit is $E \rightarrow \infty$, and $\langle N \rangle \rightarrow \infty$ with T fixed.

$$\Pi_N(f) = \lim_{K \rightarrow \infty, f \text{ fix}} B_{N,K}(f) = \frac{a^N}{N!} e^{-a} \quad (10)$$

with $a = \langle N \rangle = Kf/(1+f)$.

The 1-PTL energy distribution becomes Boltzmannian

$$P_1^{\text{eff}}(\omega) = \sum_{N=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^N \Pi_N(f) = e^{-a\omega/E} \quad (11)$$

again $T = E/\langle N \rangle$.

Approximate Tsallis–Pareto

ideal gas, general N-fluctuations

The effective 1-PTL energy expanded for $\omega \ll E$:

$$P_1^{\text{eff}}(\omega) = \sum_{N=0}^{\infty} P_N \left(1 - \frac{\omega}{E}\right)^N = 1 - \frac{\langle N \rangle}{E} \omega + \frac{\langle N(N-1) \rangle}{E^2} \frac{\omega^2}{2} + \dots \quad (12)$$

and compared to the Tsallis distribution of it:

$$P_1^{\text{TP}}(\omega) = \left(1 + (q-1) \frac{\omega}{T}\right)^{-1/(q-1)} = 1 - \frac{\omega}{T} + \frac{q}{2} \frac{\omega^2}{T^2} + \dots \quad (13)$$

Conclusion:

subleading in $\omega \ll E$

$$T = \frac{E}{\langle N \rangle}, \quad q = \frac{\langle N(N-1) \rangle}{\langle N \rangle^2} = 1 - \frac{1}{\langle N \rangle} + \frac{\Delta N^2}{\langle N \rangle^2}. \quad (14)$$

General system, general fluctuations



subleading expansion

$$\begin{aligned} \left\langle \frac{\Omega_N(E-\omega)}{\Omega_N(E)} \right\rangle &= \left\langle e^{S(E-\omega)-S(E)} \right\rangle \approx \left\langle e^{-\omega S'(E) + \omega^2 S''(E)/2 + \dots} \right\rangle \\ &= 1 - \omega \langle S'(E) \rangle + \frac{\omega^2}{2} \langle S'(E)^2 + S''(E) \rangle + \dots \end{aligned} \quad (15)$$

Comparing with Tsallis expansion interprets the parameters as

subleading in $\omega \ll E$

$$\frac{1}{T} = \langle S'(E) \rangle, \quad q = 1 - \frac{1}{C} + T^2 \Delta\beta^2 \quad (16)$$

Here $\langle S'(E) \rangle = 1/T$, $\langle S''(E) \rangle = -1/CT^2$ and $\langle S'(E)^2 \rangle = 1/T^2 + \Delta\beta^2$.

Design $q_K = 1$

by using appropriate $K(S)$

$$\begin{aligned} \left\langle e^{K(S(E-\omega)) - K(S(E))} \right\rangle &= 1 - \omega \left\langle \frac{d}{dE} K(S(E)) \right\rangle \\ &+ \frac{\omega^2}{2} \left\langle \frac{d^2}{dE^2} K(S(E)) + \left(\frac{d}{dE} K(S(E)) \right)^2 \right\rangle \end{aligned} \quad (17)$$

Note $\frac{d}{dE} K(S(E)) = K' S'$ and $\frac{d^2}{dE^2} K(S(E)) = K'' S'^2 + K' S''$.

FormLog entropy $K(S)$

Tsallis parameters T_K, q_K

Using a universal $K(S)$ FormLog we get

$$\begin{aligned}\frac{1}{T_K} &= K' \frac{1}{T} \\ \frac{q_K}{T_K^2} &= (K'' + K'^2) \left(\frac{1}{T^2} + \Delta\beta^2 \right) - K' \frac{1}{CT^2}.\end{aligned}\quad (18)$$

Useful notations: $F = 1/K' = T_K/T$ (then $F(0) = 1$) and $T^2\Delta\beta^2 = \lambda/C$.

$$q_K = \left(1 + \frac{\lambda}{C} \right) (1 - F') - \frac{1}{C} F.\quad (19)$$



DiffEq for the FormLog

solve general $q_K = 1$

$$(\lambda + C)F' + F = \lambda + C(1 - q_K) = 1 + C(q - q_K). \quad (20)$$

With $q_K = 1$ one solves $(\lambda + C)F' + F = \lambda$.

Case $\lambda = 0$ (no reservoir fluctuations): $\frac{K''}{K'} = \frac{1}{C}$.

Finite resvoir effects $1/C$ are encoded in K'' non-additivity

Set $q_K = 1$

UTI principle, $\lambda = 0$

With C independent of S we had $q = 1 - 1/C$ and obtain

$$K(S) = C \left(e^{S/C} - 1 \right) \quad (21)$$

Now, repeating subdivisions of a big set, one arrives at

$$K(S) = \sum_i p_i K(-\ln p_i). \quad (22)$$

For the above $K(S)$ is Tsallis entropy (additive), S is Rényi entropy (non-additive)

$$K(S) = \frac{1}{1-q} \sum_i (p_i^q - p_i), \quad S = \frac{1}{1-q} \ln \sum_i p_i^q. \quad (23)$$



Set $q_K = 1$

solution for constant C and λ

Define $\mu = C + \lambda$. Then

$$\lambda K'^2 - K' + \mu K'' = 0. \quad (24)$$

Solution

$$K(S) = \frac{\mu}{\lambda} \ln \left(1 - \lambda + \lambda e^{S/\mu} \right). \quad (25)$$

FormLog as double deformation

$$K(S) = h_{\mu/\lambda}^{-1} (h_{\mu}(S)) \quad \text{with} \quad h_A(S) = A \left(e^{S/A} - 1 \right) \quad (26)$$

Set $q_K = 1$

with C and λ constant, $\mu = C + \lambda$

Finally we arrive at

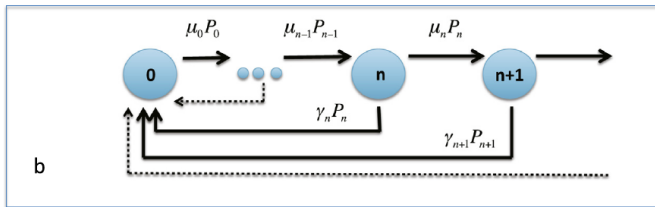
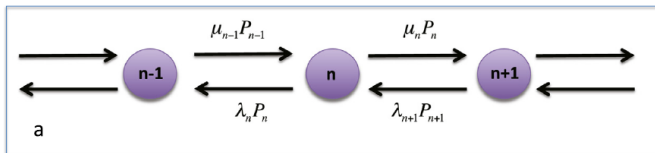
$$K(S) = \frac{\mu}{\lambda} \sum_i p_i \ln \left(1 - \lambda + \lambda p_i^{-1/\mu} \right) \quad (27)$$

Boltzmann Gaussian fluctuations ($\lambda = 1$ irresp. μ): $K(S) = - \sum_i p_i \ln p_i$.

Tsallis No fluctuations ($\lambda = 0, \mu = C$): $K(S) = C \sum_i \left(p_i^{1-1/C} - p_i \right)$.

Lambert W Extreme fluctuations ($\lambda = \mu \rightarrow \infty$): $K(S) = \sum_i p_i \ln(1 - \ln p_i)$

Diffusion and LGGR Scheme



LGGR master equation

discrete version

$$\dot{P}_n = \mu_{n-1}P_{n-1} + \delta_{n,0} \langle \gamma \rangle - (\mu_n + \gamma_n)P_n \quad (28)$$

Stationary distribution Q_n for $n \geq 1$ satisfies

$$0 = \mu_{n-1}Q_{n-1} - (\mu_n + \gamma_n)Q_n \quad (29)$$

Solution:

$$Q_n = \frac{\mu_0 Q_0}{\mu_n} \prod_{j=1}^n \left(1 + \frac{\gamma_j}{\mu_j} \right)^{-1} \quad (30)$$

LGGR master equation

particular cases 1

Simplest model: $\mu_n = \mu, \gamma_n = \gamma$ **state independent rates**

$$Q_n = Q_0 e^{-n \ln(1 + \gamma/\mu)} \quad (31)$$

"Temperature" factor: $1/T = \ln(1 + \gamma/\mu)$

For rare resets $\gamma \ll \mu$: $\mu = T\gamma$ (fluct.-diss.)

Generalized fluctuation dissipation (Einstein-Kubo) formula:

$$\mu_n = \frac{1}{Q_n} \sum_{j=n+1}^{\infty} \gamma_j Q_j. \quad (32)$$

LGGR master equation

particular cases 2

Next simplest: $\mu_n = \sigma(n + b), \gamma_n = \gamma$ **linearly preferential growth**

$$Q_n = \frac{\gamma}{\gamma + b\sigma} \frac{(b)_n}{(b + 1 + \gamma/\sigma)_n} \quad (33)$$

With the Pochhammer symbol $(b)_n = b(b + 1) \cdots (b + n - 1)$.

No "temperature" here!

Waring distribution, power-law tailed asymptotics:

$$Q_n \rightarrow \frac{\gamma}{\gamma + b\sigma} \frac{\Gamma(b + 1 + \gamma/\sigma)}{\Gamma(b)} n^{-1-\gamma/\sigma} \quad (34)$$

LGGR master equation

particular cases 3

For $\gamma_n = \sigma(n - a)$ and $\mu_n = \sigma \frac{a}{k}(n + k)$

$$Q_n = \binom{n+k-1}{n} \frac{a^n k^k}{(a+k)^{n+k}} \quad (35)$$

the stationary solution is an NBD with $\langle n \rangle = a$.

$k = 1$ case: $Q_n = (1 - q)q^n$, geometrical distribution.

$k = \infty$ case: $Q_n = \frac{a^n}{n!} e^{-a}$, Poisson.

LGGR master equation

NBD interpretation

State n : having n hadrons, a new is created with rate μ_n .

Reset: a collective re-melting into QGP (or overlap of wide resonances)

- critical $n = a$: less hadrons lead to $\gamma_n < 0 = \text{QGP} \rightarrow n$ hadrons.
- $n = 0$: QGP is created with rate $-\gamma_0 = \sigma a$.
- μ_n growth rate by one: k scales the Matthew principle, how much n hadrons assist to create a further one.
- $\gamma_0 + \mu_0 = 0$: no hadron loss rate from zero hadrons.

LGGR master equation

continuous version



$$\frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} (\mu(x)P(x, t)) - \gamma(x)P(x, t) \quad Q(x) = \mu(0)Q(0) e^{-\int_0^x \frac{\gamma(t)+\mu'(t)}{\mu(t)} dt}$$

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Table 1

Resetting and growth rates for the most common stationary PDF-s.

$\gamma(x)$	$\mu(x)$	$Q(x)$
γ	μ	Exponential: $\sim e^{-(\gamma/\mu)x}$
γ	$\sigma(x+b)$	Tsallis–Pareto: $\sim (1+x/b)^{-1-\gamma/\sigma}$
γ	$\sigma x^\alpha, \alpha < 1$	Weibull: $\sim x^{-\alpha} e^{-bx^{1-\alpha}}$
γ	$\sigma(x+a)(x+b)$	Pearson: $\sim (x+a)^{-1-\nu}(x+b)^{-1+\nu}$
γ	σe^x	Gompertz: $\sim \exp\left(\frac{\gamma}{\sigma} e^{-x} - x\right)$
$\ln(x/a)$	σx	Log-Normal: $Q(x) dx \sim e^{-\gamma^2/2\sigma} d\gamma$
x	σ^2	Gauss: $\sim e^{-x^2/2\sigma^2}$
$\sigma(ax-c)$	σx	Gamma: $\sim x^{c-1} e^{-ax}$

Further LGGR features

convergence speed estimate

- 1 Define entropic divergence with a $\kappa(\xi) \geq 0$ fct. of $\xi = P(x, t)/Q(x)$:

$$\rho[P, Q] \equiv \int_0^{\infty} \kappa(\xi) Q dx \geq 0$$

- 2 Based on \dot{P} look for $\dot{\xi}$ and $\dot{\kappa}$. Note: $\kappa(1) = 0$, and fix $\xi(0, t) = 1$ boundary.

- 3 Conclude that $\dot{\rho} = - \int_0^{\infty} \kappa \gamma Q dx \leq 0$.

- 4 Using Jensen inequality with $p(x) = \gamma Q / \langle \gamma \rangle_{\infty}$ get a limit on the minimal speed to Q as

$$\dot{\rho} \leq - \langle \gamma \rangle_{\infty} \kappa \left(\frac{\langle \gamma \rangle_t}{\langle \gamma \rangle_{\infty}} \right).$$

Brief Summary

- Tsallis–Pareto distribution is natural
- It is the next to simplest (including the simplest)
- Modern stat.phys. models are relevant in high-energy physics

Outlook

- Colleague 1: Good.
- Colleague 2: Even Better.
- Enemy: *Wrong!*
- The Last Question: if the whole universe exists due to entropy (Hawking, Bekenstein, Verlinde, etc) due to which entropy?

formal entropy

formal cocktail

